A Crash Course on Discrete Stability and Finite Element Methods

Leszek Demkowicz
Institute for Computational Engineering and Science (ICES)
The University of Texas at Austin

Abstract

Finite Element (FE) Method is a special case of Galerkin method and, in this sense, it draws its origins directly from Ritz and Galerkin. In this course, intended for non-specialists, we focus on explaining the convergence mechanisms behind various versions of Galerkin and FE methods taking you along a historical path.

1 Day 1: Variational Formulations

1. Elastic bar example.
2. Classical calculus of variations, 1D example, variational formulation.
3. Coping with a non-homogeneous essential boundary condition, concept of a (finite energy) lift.
4. Abstract variational formulation.
5. Various variational formulations for a model diffusion-convection-reaction problem.

2 Day 2: Ritz and Galerkin Methods

1. Minimization of the total energy functional, equivalence of minimization and variational problems.
2. Ritz and Galerkin methods.
3. Internal (conforming) discretizations - what does it mean in practice?
4. H^1, $H(\text{curl})$, $H(\text{div})$ and L^2 energy spaces, different conformity conditions.
5. Lax-Milgram Theorem and Cea's Lemma, coercivity.

3 Day 3: Interpolation in Sobolev Spaces. Mikhlin's Theory

1. Degrees of freedom and interpolation operator.
2. Projection Based (PB) interpolation.
3. Non-coercive problems: vibrations, wave propagation.

4. Mikhlin's theory of asymptotic stability and convergence.
5. Linear elasticity.

4 Day 4: Babuška's Theorem

1. Well posedness of a general variational problem: Banach-Babuška-Nečas Theorem.
2. Babuška Theorem and concept of discrete stability.
3. Mixed problems and Brezzi's theory.
4. SUPG method of Hughes and stabilized methods.

5 Day 5: DPG Method with Optimal Test Functions

1. Three interpretations of the DPG method: Petrov-Galerkin method with optimal test functions, minimum residual method, mixed method.
2. Broken test spaces.
3. How to implement the DPG method.
4. Examples of applications: elasticity, wave propagation problems (including Maxwell equations), compressible and incompressible NS equations.
5. Double adaptivity method of Dahmen.

Prerequesities:

Elementary continuum mechanics (linear elasticity, beams, plates, vibrations), elementary functional analysis (concept of vector space, norm, inner product), elementary differential and integral calculus (directional and Gâteaux derivatives, integration by parts, grad,curl,div operators)

Format

There will be a total of 6 hours of lectures and discussion per day: 3 hours of lectures before lunch, 1h lecture after lunch and less formal 2h discussion session after lunch, including presentation of numerical results, computer codes and discussion of homework problems.