Mathematical Theory of Finite Elements
(A crash course for engineers)

Leszek Demkowicz

Institute for Computational Engineering and Sciences (ICES)
The University of Texas at Austin

Abstract

We review fundamentals of Galerkin and conforming Finite Element (FE) methods using the model diffusion-convection-reaction problem. We discuss the possibility of different variational formulations leading to different energy spaces and corresponding conforming elements. The course is focusing on the famous inf-sup stability condition and the concept of discrete stability. We review the classical results of Babuśka, Mikhlin and Brezzi, and finish the exposition with fundamentals of the Discontinuous Petrov Galerkin (DPG) method. The week-long course consists of three 1.5 hour lectures per day accompanied with a one hour afternoon Q/A discussion session.

Day 1

1. Classical calculus of variations. Concept of a variational formulation.
2. Diffusion-convection-reaction model problem. Different variational formulations.
3. Distributional derivatives and different energy spaces.

Day 2

1. Abstract framework: vector space, linear and bilinear forms, dual space.
2. Galerkin and Riesz methods.
3. Exact sequence elements.

Day 3

1. Banach Closed Range, Babuška-Nečas, and Babuška Theorems.
2. Coercivity. Lax-Milgram Theorem and Cea's Lemma.
3. Well posedness of the variational formulations for the model problem.

Day 4

1. Mikhlin's theory of asymptotic stability and convergence.
2. Brezzi's theory of mixed problems.
3. Concept of optimal test functions.

Day 5

1. Breaking test spaces and bilinear forms.
2. Fundamentals of the Discontinuous Petrov-Galerkin (DPG) Method.
3. Current research on the DPG method.

References

[1] L. Demkowicz. Various variational formulations and Closed Range Theorem. Technical report, ICES, January 15–03.

[2] F. Fuentes, B. Keith, L. Demkowicz, and S. Nagaraj. Orientation embedded high order shape functions for the exact sequence elements of all shapes. *Comput. Math. Appl.*, 70:353–458, 2015.

Exercises

1. Derive the variational formulation and the corresponding Euler-Lagrange boundary-value problem for the two-dimensional minimization problem:

$$\begin{cases} u = u_0 \text{ on } \Gamma_1 \\ \displaystyle\int_\Omega F(x, y, u(x, y), \frac{\partial u}{\partial x}(x, y), \frac{\partial u}{\partial y}(x, y)) \; dxdy \to \min \end{cases}$$

Here $\Omega \subset \mathbb{R}^2$ is a bounded two-dimensional domain with boundary Γ split into two disjoint parts, $\Gamma = \Gamma_1 \cup \Gamma_2$. (5 points)

2. (An interface problem) Consider the elastic beam pictured in Fig. 1. Deflection $w(x)$ of the beam minimizes the *total potential energy* given by the functional

$$J(w) = \frac{1}{2}\int_0^{3l/2} EI(w'')^2 - \left[\int_0^{3l/2} qw + P_0 w(\frac{3l}{2}) + M_0 w'(\frac{3l}{2})\right]$$

among all possible displacements that satisfy the *kinematic BC*:

$$w(0) = w'(0) = w(l) = 0$$

- Derive the Gâteaux derivative of cost functional $J(w)$ and the corresponding variational formulation for the problem.
- Use integration by parts (twice) and the Fourier's Lemma argument to derive the corresponding E-L equation(s) in subintervals $(0, l)$ and $l, 3l/2)$, boundary conditions at $x = 3l/2$ and interface conditions at $x = l$.
- Show the (formal) equivalence between the variational formulation and the E-L interface boundary-value problem.

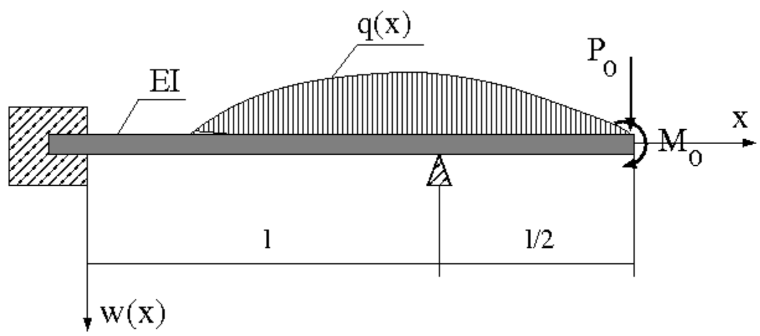

Figure 1: An elastic beam example

(7 points)

3. Integration by parts formulas. Let $\Omega \subset \mathbb{R}^3$ be a domain with boundary $\partial\Omega$. Use elementary integration by parts to derive the following integration by parts formulas.

$$\int_\Omega \nabla u\, v = -\int_\Omega u\, \nabla v + \int_{\partial\Omega} n u\, v$$
$$\int_\Omega (\nabla \times E) \cdot F = \int_\Omega E \cdot (\nabla \times F) + \int_{\partial\Omega} (n \times E) \cdot F$$
$$\int_\Omega (\nabla \cdot u)\, v = -\int_\Omega u \cdot (\nabla v) + \int_{\partial\Omega} u \cdot n\, v$$

(5 points)

4. Formulate the classical diffusion-convection-reaction problem in terms of a second order PDE:

$$-\frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u}{\partial x_j} - b_i(x)u) + c(x)u = f(x)$$

with two types of boundary conditions (BCs):

$$u = u_0 \quad \text{and} \quad \sigma \cdot n = \sigma_0 \cdot n\,,$$

where $\sigma_i = a_{ij}\frac{\partial u}{\partial x_j} - b_i(x)u$ denotes the total flux, u_0, σ_0 are given, and n denotes the outward normal. Derive the classical variational formulation and identify the corresponding functional setting: energy trial and test space, bilinear and linear forms. (5 points)

5. Replace the second order diffusion-convection-reaction equation with a system of first order equations. Use the same BCs as in the previous problem. Derive then (formally, no math details expected) the corresponding *six* variational formulations and identify the corresponding functional settings. Identify the (group) unknown, the (group) test function, and the bilinear and linear forms. Explain what we mean by a *symmetric functional setting* ? (15 points)

6. Let $\Omega \subset \mathbb{R}^N$, $N = 2, 3$ be a bounded domain (i.e. open and connected set) with boundary Γ split into two disjoint parts Γ_p and Γ_u. Consider the first order system of linear time-harmonic acoustics equations:

$$\begin{cases} i\omega p + \nabla \cdot u &= f \\ i\omega u + \nabla p &= g \end{cases}$$

accompanied with BC:

$$\begin{aligned} p &= p_0 && \text{on } \Gamma_p \quad \text{(soft BC)} \\ u \cdot n &= u_0 \cdot n && \text{on } \Gamma_u \quad \text{(hard BC).} \end{aligned}$$

Above, p is the pressure, u stands for the velocity, ω denotes the angular velocity, i is the imaginary unit, and n is the outward unit vector. Functions f, g define sources and, similarly, $p_0, u_0 \cdot n$ denote the prescribed pressure and normal velocity on the boundary.

What physical principles do the two differential equations represent ? Follow the discussion in class to derive the four possible variational formulations and the corresponding two reduced formulations. Identify the energy spaces. Explain which formulations enjoy a symmetric functional setting and which do not. (15 points)

7. Repeat the same discussion for the system of time-harmonic Maxwell equations:

$$\begin{cases} \frac{1}{\mu}\nabla \times E = -\omega H & \text{(Faraday's Law)} \\ \nabla \times H = J^{\text{imp}} + \sigma E + i\omega E & \text{(Ampère's Law)} \end{cases}$$

accompanied with BC:

$$\begin{aligned} n \times E &= n \times E_0 && \text{on } \Gamma_E \\ n \times H &= n \times H_0 && \text{on } \Gamma_H \end{aligned}$$

where boundary has been split into two disjoint parts Γ_E and Γ_H. Above E is the electric field, H is the magnetic field, ϵ, μ, σ are the material constants: permittivity, permeability and conductivity, J^{imp} is a given (impressed) electric field, and $n \times E_0, n \times H_0$ are given BC data. Follow the discussion in class to derive the four possible variational formulations and the corresponding two reduced formulations. Identify the energy spaces. Explain which formulations enjoy a symmetric functional setting and which do not. (20 points)

8. Write down *static isotropic elasticity* equations in the form of a first order system: constitutive equations combined with Cauchy's definition of strain, and equilibrium (linear momentum) equations ? What about the angular momentum equations, are they satisfied as well ? Derive the corresponding various variational formulations and identify the functional setting. How many of them do we have ? Is it the same number as for the diffusion-convection-reaction problem (six) ? (20 points)

9. Discuss intuitively problem with the volumetric locking. Which of the variational formulations for the linear elasticity are expected to be stable *uniformly* in Poisson ratio ν and, therefore, the corresponding FE discretizations will not "lock" ? (10 points)

10. Discuss changes in variational formulations from Problems 3,4 when the original BCs are replaced with the Cauchy condition,
$$\sigma \cdot n - \beta u = g$$
where $\beta > 0$ is a given constant, and g is given. (20 points)

11. Equivalence of continuity and boundedness for linear(antilinear) forms. Let V be a normed vector space and l be a linear (antilinear) functional defined on V. Prove that the following conditions are equivalent to each other. (5 points)

    (i) l is continuous on V,

    (ii) l is continuous at 0 (zero vector),

    (iii) l is *bounded*, i.e. there exists $C > 0$ such that
    $$|l(v)| \le C\|v\|_V$$
    where $\|v\|_V$ is the norm in V.

12. Equivalence of continuity and boundedness for bilinear(sesquilinear) forms. Let U, V be normed vector spaces and b be a bilinear (sesquilinear) functional defined on $U \times V$. Prove that the following conditions are equivalent to each other. (5 points)

    (i) b is continuous on $U \times V$,

    (ii) b is continuous at $(0, 0)$,

    (iii) b is *bounded*, i.e. there exists $M > 0$ such that
    $$|b(u, v)| \le M\|u\|_U \, \|v\|_V \, .$$

13. Dual norm. Let V be a normed vector space and l be a continuous (bounded) linear (antilinear) functional defined on V. Let $\|l\|$ be the "smallest" constant that we can use in the boundedness condition,
$$\|l\| := \inf\{C \,:\, |l(v)| \le C\|v\|_V\}$$

(a) Prove equivalent characterizations for $\|l\|$,

$$\|l\| = \sup_{v \neq 0} \frac{|l(v)|}{\|v\|} = \sup_{\|v\|=1} |l(v)|$$

(b) Let V' be the collection of all bounded linear (antilinear) functionals defined on V. Argue that V' is close wrt the standard operations on functions and, therefore, constitutes a subspace of algebraic dual V^* consisting of all linear (antilinear) functionals on V. Prove that $\|l\|$ satisfies the axioms for a norm, i.e V' is a normed space (called the *topological dual* of space normed space V).

(10 points)

14. Ritz method. Assume $b(u, v)$ is symmetric and positive-definite. Let $U_h \subset U$ be a finite-dimensional subspace of energy space U. Define the total potential energy functional as

$$J(u) = \frac{1}{2} b(u, u) - l(u)$$

where $l \in U'$ is a linear continuous functional defined on U. Prove that the following problems are equivalent to each other. (20 points)

- Minimization of energy over the approximate space U_h:

$$J(u_h) = \min_{w_h \in U_h} J(w_h)\,.$$

- Galerkin approximation of the variational problem:

$$\begin{cases} u_h \in U_h \\ b(u_h, v_h) = l(v_h) \quad \forall v_h \in U_h\,. \end{cases}$$

- Minimization of the distance between the exact and approximate solutions in the energy norm:

$$\|u - u_h\|_E = \min_{w_h \in U_h} \|u - w_h\|_E$$

  where $\|v\|_E^2 := b(v, v)$.

- Minimization of the residual in the norm dual to the energy norm,

$$\sup_{v \in U} \frac{|b(u_h, v) - l(v)|}{\|v\|_E} = \min_{w_h} \sup_{v \in U} \frac{|b(w_h, v) - l(v)|}{\|v\|_E}$$

15. Prove Cea's lemma. Consult other sources, if necessary. What was the main advancement of Cea's result when compared with Ritz method ? (10 points)

16. Discuss in your own words Babuška's Theorem and explain the origin of the famous phrase: *discrete (uniform) stability and approximability imply convergence*. (10 points)

17. Prove Babuška's Theorem. (20 points)

18. Explain Mikhlin's concept of *asymptotic stability*. (5 points)

19. Write in your own words what have you understood and retained about the DPG method. (10 points)

20. Distributional derivatives. Let a domain $\Omega \subset \mathbb{R}^N$, $N = 2, 3$, be split into two subdomains Ω_1, Ω_2 with a smooth interface Γ. Let u, E, v be functions consisting of two smooth branches u^I, E^I, v^I, $I = 1, 2$ defined in the subdomains. By "smooth" we understand $u^I \in C^1(\overline{\Omega_I})$ etc. Let n be the unit vector on interface Γ pointing from subdomain Ω_1 into subdomain Ω_2.

    (i) Let $\phi \in C_0^\infty(\Omega)$ be a Schwartz test function (scalar- or vector-valued). Use elementary integration by parts to derive the following formulas:

    $$-\int_\Omega u \nabla\phi = \sum_I \int_{\Omega_I} \nabla u^I \, \phi + \int_\Gamma [u] n \phi \,,$$
    $$\int_\Omega E \, \nabla \times \phi \,, = \sum_I \int_{\Omega_I} \nabla \times E^I \, \phi + \int_\Gamma [n \times E] \phi \,,$$
    $$\int_\Omega v \, \nabla \cdot \phi = \sum_I \int_{\Omega_I} \nabla \cdot v^I \, \phi + \int_\Gamma [n \cdot v] \phi$$

    where

    $$[u] = u^2 - u^1, \quad [n \times E] = n \times (E^2 - E^1), \quad [n \cdot v] = n(v^2 - v^1) \,.$$

    (ii) Interpret the formulas above in the language of distributions using the definition of regular distributions, distributional derivatives and corresponding operators of grad, curl and div understood in the distributional sense. You will have to introduce a multidimenional equivalent of Dirac's delta.

    (iii) Conclude that functions u, E, v belong to energy spaces $H^1(\Omega), H(\text{curl}, \Omega), H(\text{div}, \Omega)$ *if and only if* the corresponding continuity conditions across the interface Γ are satisfied:

    $$[u] = 0, \quad [n \times E] = 0, \quad [n \cdot v] = 0 \,.$$

    (20 points)

21. Affine coordinates. Prove the following facts about the affine coordinates:

    - The affine coordinates are independent of the enumeration of vertices (in the presented construction, we considered vectors $x - a_0, a_i - a_0$, $i = 1, 2, 3$, so it looks like things might depend upon the choice of vertex a_0).
    - The *affine coordinates are invariant under affine transformations*: if λ_i are affine coordinates of a point x with respect to vertices a_i then λ_i are also affine coordinates of a point Tx with respect to vertices Ta_i, for any bijective affine map T:

    $$Tx := a + Ax$$

where $a \in \mathbb{R}^3$, and A is a non-singular 3×3 matrix.

- In 2D, the affine coordinates may be interpreted as *area coordinates*. Prove that

$$\lambda_i = \frac{\text{area of } T_i}{\text{area of } T}, \quad i = 0, 1, 2$$

where subtriangles T_i of triangle T are defined in Fig. 2.

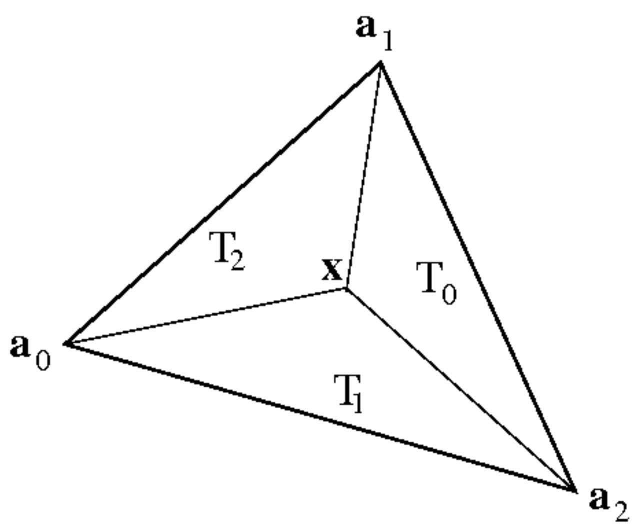

Figure 2: Area coordinates.

Be concise. (10 points)

22. Whitney shape functions. Write down formulas for the Whitney shape functions in terms of affine coordinates and their gradients. Discuss their vanishing properties and explain how you "glue" them to obtain Galerkin basis functions for the energy spaces forming the exact sequence. Use [2] if necessary.

    (10 points)

23. Shape functions for the lowest order hexahedron. Write out shape functions for the lowest order hexahedron in terms of 1D affine coordinates and their derivatives. Use [2] if necessary.

    (10 points)

24. Characterization of Nedelec's space. Let $\tilde{\mathcal{P}}^k$ denote homogeneous polynomials of order k. Prove the following identity.

$$x \times (\tilde{\mathcal{P}}^{p-1})^3 = \{E \in (\tilde{\mathcal{P}}^p)^3 \,:\, x \cdot E(x) = 0 \quad \forall x\}$$

    (10 points)

When necessary, consult the notes on my web page:

`http://users.ices.utexas.edu/~leszek/classes.html`

(Exercises for Advanced Theory of Finite Element Methods (EM394H/CAM394H)).